Ex. Show that the set $S=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

→ a basis for $M_{2\times2}$

i) S spans $M_{2\times2}$

$$\begin{pmatrix}4&6\\-2&-3\end{pmatrix}=4\begin{pmatrix}1&0\\0&0\end{pmatrix}+6\begin{pmatrix}0&1\\0&0\end{pmatrix}+(-2)\begin{pmatrix}0&0\\1&0\end{pmatrix}+(-3)\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

ii) $$c_1\begin{pmatrix}1&0\\0&0\end{pmatrix}+c_2\begin{pmatrix}0&1\\0&0\end{pmatrix}+c_3\begin{pmatrix}0&0\\1&0\end{pmatrix}+c_4\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$\therefore c_1=c_2=c_3=c_4=0$ Linearly independent

$\therefore$ S is a basis for $M_{2\times2}$

5] The dimension of a vector space

Def. If a vector space V has a basis consisting of n vectors, then the number n is called the dimension of V denoted by $\dim(V)=n$

eg. $\dim(M_{2\times2})=4$ (4 vectors in basis of $M_{2\times2}$)

eg. $\dim(\mathbb{R}^3)=3$ (3 vectors in basis of $\mathbb{R}^3$)

eg. $\dim(P_n)=n+1$

eg. $\dim(M_{m\times n})=m\times n$

eg. $\dim(\mathbb{R}^n)=n$

} General

The rank of a matrix

Row and Column space of a matrix

Let A be an $m\times n$ matrix

① The row space of A is the space spanned by the row vectors of A

② The column space of A is the space spanned by the column vectors of A

Ex. $A=\begin{pmatrix}3&2&0\\8&7&-4\end{pmatrix}$

Row space $\{(3,2,0)\ (8,7,-4)\}\in\mathbb{R}^3$

Colums pace $\{(3,8)^T\ (2,7)^T\ (0,-4)^T\}\in\mathbb{R}^2$

Row vector $\{(3,2,0)\ (8,7,-4)\}$

~~Col~~column vector $\left\{\begin{pmatrix}3\\8\end{pmatrix}\begin{pmatrix}2\\7\end{pmatrix}\begin{pmatrix}0\\-4\end{pmatrix}\right\}$

Ex. $A = \begin{pmatrix} 10 & 0 & 5 & 4 \\ 0 & 2 & -7 & 1 \\ 3 & 1 & 2 & 1 \end{pmatrix}$

Row space $\{(10,0,5,4)\ (0,2,-7,1)\ (3,1,2,1)\} \in \mathbb{R}^4$

Column space $\{(10,0,3)^T\ (0,2,1)^T\ (5,-7,2)^T\ (4,1,1)^T\} \in \mathbb{R}^3$

How to find the basis for a row space of a matrix?

Theorem: If a matrix A is row equivalent to another matrix B in row echelon form, then the non-zero rows of B form ~~the~~ a basis for the row space of A.

Ex. Find the basis for the row space of

$$A = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \text{Basis } \{(1,3,1,3)\ (0,1,1,0)\ (0,0,0,1)\}$$

$\text{rank}(A) = 3$

Def. The rank of a matrix A is the dimension of the row space of the matrix A and is denoted by rank(A).

Ex. Find the rank of A

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 5 & 5 & -5 \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 5 & 5 & -5 \end{pmatrix}$$

$$\xrightarrow{R_3 - 5R_2} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -10 & -30 \end{pmatrix}$$

Basis $\{(1,-2,0,1)\ (0,1,3,5)\ (0,0,-10,-30)\}$

$\text{rank}(A) = 3$

Rank of matrix and system of linear equations

[1] System of non-homogeneous equations $(AX = B)$

$Ax = B$

- $\text{rank}(A \vdots B) = \text{rank}(A) \Rightarrow$ consistent
  - $\text{rank}(A) =$ no. of unknowns $= n \Rightarrow$ unique solution
  - $\text{rank}(A) < n \Rightarrow$ infinite solutions with no. of parameters $= n - \text{rank}(A)$
- $\text{rank}(A \vdots B) \neq \text{rank}(A) \Rightarrow$ not consistent $\equiv$ no solution

Solve:

$$x_1 \qquad - 2x_3 + x_4 = 5$$
$$3x_1 + x_2 - 5x_3 \qquad = 8$$
$$x_1 + 2x_2 \qquad - 5x_4 = -9$$

$$(A \vdots B) = \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 2 & 2 & -6 & -14 \end{array}\right) \xrightarrow{R_3 - 2R_2}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\text{rank}(A) = 2$
$\text{rank}(A \vdots B) = 2$ $\Rightarrow$ Consistent

$\text{rank}(A) = 2 < 4 \Rightarrow$ infinite no. of solutions

let $x_3 = s$, $x_4 = t$

$x_2 = -7 - s + 3t$

$x_1 = 5 + 2s - t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 + 2s - t \\ -7 - s + 3t \\ s \\ t \end{pmatrix}$$

I System of homogeneous equations $(Ax=0)$

$$Ax=0 \begin{cases} \text{rank}(A) = n \text{ (no. of unknowns)} \Rightarrow \text{trivial solution} \\ \text{rank}(A) < n \Rightarrow \text{infinite solutions} \end{cases}$$

Ex. Solve

$$3x_1 - 6x_2 - x_3 - x_4 = 0$$
$$x_1 - 2x_2 + 5x_3 - 3x_4 = 0$$
$$2x_1 - 4x_2 + 3x_3 - x_4 = 0$$

$$A = \begin{pmatrix} 3 & -6 & -1 & -1 \\ 1 & -2 & 5 & -3 \\ 2 & -4 & 3 & -1 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

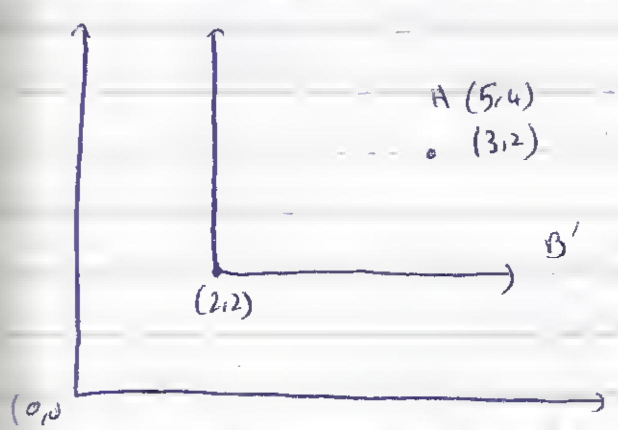
rank $= 3 < 4$
infinite solutions

let $x_2 = t$

$x_1 = 2t$, $x_3 = 0$, $x_4 = 0$

Coordinate System and change of basis

e.g. $\mathbb{R}^2 \Longrightarrow$ basis $B = \{(0,1)\ (1,0)\}$



$H = (3,2)_{B'}$

Def: Let $B = \{v_1, v_2, \dots, v_n\}$ to be basis for a vector space V and let x be a vector in V such that $x = c_1v_1 + c_2v_2 + \dots + c_nv_n$, then the scalars $c_1, c_2, \dots, c_n$ are called the coordinates of x

Ex. $x = (5, 8, 4, -1) \in \mathbb{R}^4$

basis of $\mathbb{R}^4$ $\{(1,0,0,0)\ (0,1,0,0)\ (0,0,1,0)\ (0,0,0,1)\}$

$x = 5(1,0,0,0) + 8(0,1,0,0) + 4(0,0,1,0) - 1(0,0,0,1)$

Change of basis.

Ex. Find the coordinates of $x = (1,2,-1)$ in $\mathbb{R}^3$ relative to the basis

$B' = \{(1,0,1), (0,-1,2)\ (2,3,-5)\}$

$x = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5)$

$\cancel{c_1 + 2c_3 = 1}$

$\cancel{-c_2}$

$$\begin{aligned} c_1 + 0 + 2c_3 &= 1 \\ 0 - c_2 + 3c_3 &= 2 \\ c_1 + 2c_2 - 5c_3 &= -1 \end{aligned}$$

$c_1 = 5,\ c_2 = -8,\ c_3 = 2$

$(x)_{B'} = (5, -8, 2)$

How to change basis

$$\begin{tikzcd} & x \arrow[dl, dash] \arrow[dr, dash] & \\ (x)_B & & (x)_{B'} \end{tikzcd}$$

$(x)_{B'} = P^{-1}(x)_B$

P^{-1} = transition matrix

$(x)_B$ = the general coordinates

To find the transition matrix we use row operations to make the following form $(B' \mid B) \Longrightarrow (I_n \mid P^{-1})$

Ex) $(x)_B = (1, 2, -1)$

$B = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$ $\quad$ $B' = \{(1,0,1)\ (0,-1,2)\ (2,3,-5)\}$

Find $(x)_{B'}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -5 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right)$$

$$(x)_{B'} = P^{-1}(x)_B = \begin{pmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \\ -2 \end{pmatrix}$$